Lab Report
Fourier-Optics

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1 Introduction

1.1 Fourier Optics at a glance

First off, we are going to take a short look on Fourier Optics. Compared to 'Classic Optics'[1] Fourier Optics offers a slightly different, or maybe rather a more sophisticated/extended, view to optics and therefore some new interesting aspects. Some classic aspects[2] are combined with the mathematical model of Fourier Transformations. Since Fourier Optics applys the idea of fourier transform taking place in a fourier plane, we can basicly do things like spatial filtering which means a form of optical signal processing.[3] Another rather typical thing, where fourier optics is involved is holographic imaging, where along with the intensity, phase information is recorded onto a film plate. How Fourier Optics actually works and in what way fourier transforms are involved will be discussed in the next section, where we will focus a little bit closer on the theory of fourier optics.

1.2 A deeper look at Fourier Optics

In this section we will show, that every (thin) lense will produce a fourier transform. The first step will be, to show that this is true, for the case where we use collimated light shining upon an object which is placed in the focal plane of a lense. In this case the fourier transform is visible (using a screen) in the second focal plane on the other side of the lense. The next step will be, to show, that the same is true for not properly collimated light (slighty deverging light) and the object being placed at certain positions around the focal plane of the lense. The latter one is going to be the tough part.

Scenario 1:

Collimated light, and a setup where the object ist placed in the focal plane of a lense and the fourier transform will be visible in the second focal plane.

[1]Read: Ray Model and Huygen's Wave Model as well as Maxwell's electro magnetic Model.
[2]As Inteference and diffraction in a common sense.
[3]Examples would be: picture improvement or blurring of images.

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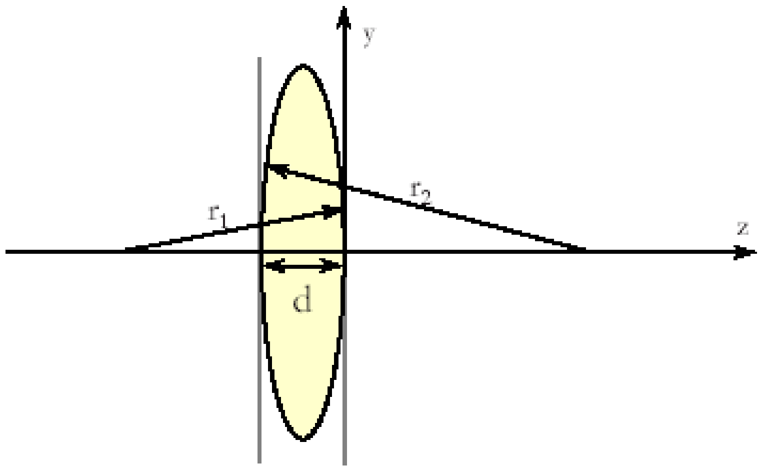


Thin lens with thickness d and refraction index of n. As we know from optics $\frac{1}{f} = (n-1)\frac{1}{r_1} + \frac{1}{r_2}$. The following is valid for the right hand surface:

$$y^2 + (z + r_1)^2 = r_1^2 \Rightarrow z = -r_1 + \sqrt{r_1^2 - y^2}$$

and left hand surface:

$$y^2 + (z - r_2 + d)^2 = r_2^2 \Rightarrow z = r_2 - d - \sqrt{r_2^2 - y^2}$$

The thickness of the glass at the height of y:

$$T(y) = -r_1 - r_2 + d + \sqrt{r_1^2 - y^2} + \sqrt{r_2^2 - y^2} \approx d - \frac{y^2}{2r_1} - \frac{y^2}{2r_2}$$

where $r_1, r_2 \gg y$.
Phase changes relative to y=0:

In air:

$$\Delta\varphi_i = -\left[d - T(y)\right]\frac{2\pi}{\lambda} = -\frac{\pi}{\lambda}\left(\frac{1}{r_1} + \frac{1}{r_2}\right)y^2$$

In glass:

$$\Delta\varphi_l = -T(y)\frac{2\pi}{\lambda'} = -\frac{2\pi n}{\lambda}\left[d - \frac{1}{2}\left(\frac{1}{r_1} + \frac{1}{r_2}\right)y^2\right]$$

Total Phase Change:

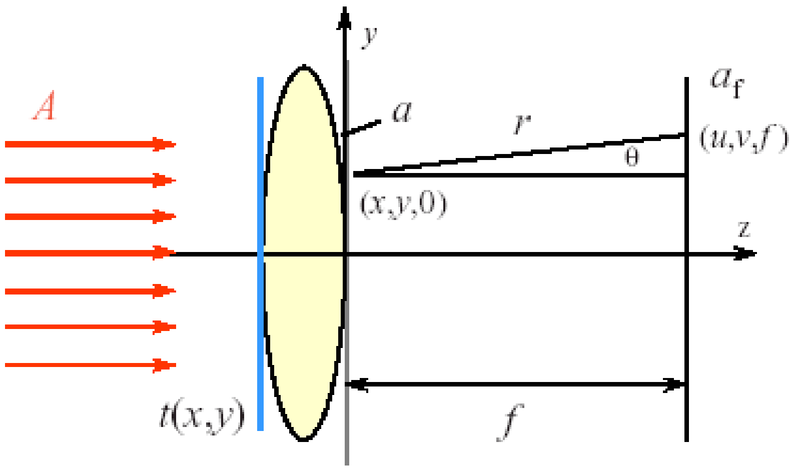
$$\Delta\varphi_y = \Delta\varphi_i + \Delta\varphi_l = -\frac{2\pi n}{\lambda}d + \frac{(n-1)\pi}{\lambda}\left(\frac{1}{r_1} + \frac{1}{r_2}\right)y^2$$

using $\frac{1}{f} = (n-1)\frac{1}{r_1} + \frac{1}{r_2}$:

$$= \frac{\pi}{\lambda f} y^2$$

And in both, x- and y-directions:

$$\Delta\varphi = \Delta\varphi_x + \Delta\varphi_y = \frac{\pi}{\lambda f}\left(x^2 + y^2\right)$$



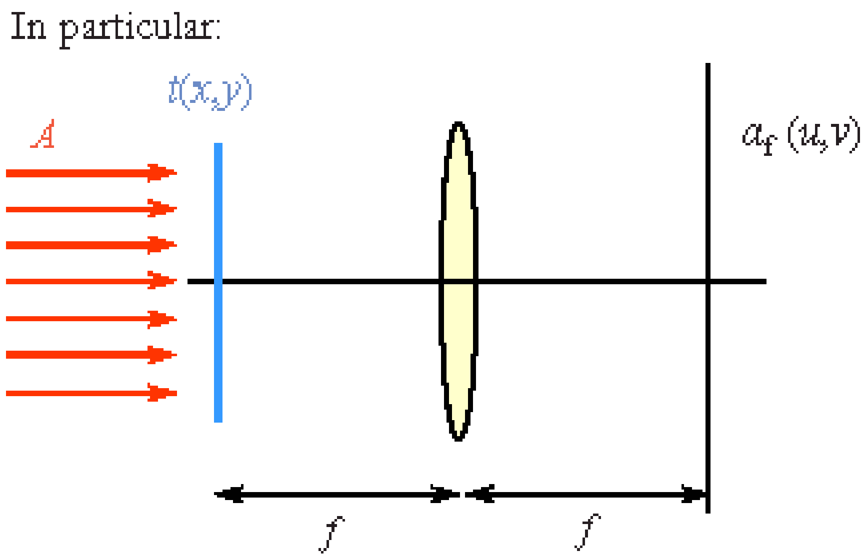
- Incident Field A (plane Wave)
- Thin Object with transmission t(x,y) in front of the lens
- Field immediately after the lens $a = At(x,y)e^{\frac{i\pi}{\lambda f}\left(x^2+y^2\right)}$

$\Delta\varphi = \frac{\pi}{\lambda f}\left(x^2 + y^2\right)$
Field in the focal plane at $z = f$
Fresnel diffraction ⇒

$$a_f(u,\nu) = \frac{i}{\lambda f}\int\int dxdy\ At(x,y)\, e^{\frac{i\pi}{\lambda f}\left(x^2+y^2\right)} e^{\frac{i2\pi}{\lambda}r}$$

$$r = \sqrt{f^2 + (u-x)^2 + (\nu-y)^2} \approx f + \frac{(u-x)^2}{2f} + \frac{(\nu-y)^2}{2f}$$

$$a_f(u,\nu) = \frac{i}{\lambda f} e^{\frac{i2\pi}{\lambda}f} e^{-\frac{i\pi}{\lambda f}\left(u^2+\nu^2\right)} \int\int dxdy\ At(x,y)\, e^{\frac{i2\pi}{\lambda f}(xu+y\nu)}$$



Assumption: A=const. Define $\xi = \frac{u}{\lambda f}$ and $\eta = \frac{\nu}{\lambda f}$

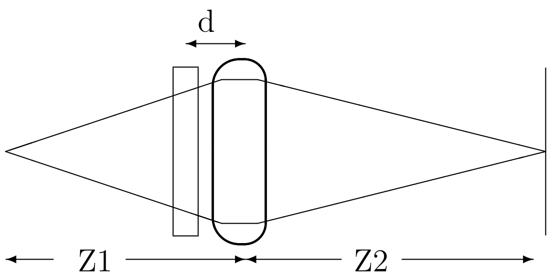
$$a_f\,(u,\nu) = Constant \;\times\; e^{-\frac{i\pi}{\lambda f}(u^2+\nu^2)} \cdot T(\nu,\eta)$$

$$T(\nu,\eta) = F\,\{t(x,y)\}$$

$$a_f(u,\nu) = Constant \;\times\; T(\xi,\eta)$$

And we conclude that a thin lens does a 2 dimensional Fourier transform.

Scenario 2:



A non-collimated beam shines upon the object, which is placed in the distance d before the lens. the lightsource is placed in the distance $z_1 > d$ in front of the object, the plane where the lightsource is imaged should be in the distance z_2.[4]

[4]which should be the fourier plane.

z_1, z_2 and the focal length f follow the law:
$z_1^{-1} + z_2^{-1} - f^{-1} = 0$. Using the operator method we get the sequence:

$$S = R\left[z_2\right] Q\left[-\frac{1}{f}\right] R\left[d\right] Q\left[\frac{1}{z_1 - d}\right]$$

Reading the operators from right to left they represent the following: furthest right Q is the deverging spherical light wave, the operator R to the left of it is the propagation over the distance d, the next operator Q stands for our lens and the R furthest to the left, well is the final propagation over the distance z_2. Taking the earlier noted equation $z_1^{-1} + z_2^{-1} - f^{-1} = 0$ we can transform $Q\left[\frac{-1}{f}\right]$ into $Q\left[-\frac{1}{z_1} - \frac{1}{z_2}\right]$.
Our next step will be to replaced the two operators to the left being R followed by Q.

$$R\left[z_2\right] Q\left[-\frac{1}{z_1} - \frac{1}{z_2}\right] = Q\left[\frac{z_1 + z_2}{z_2^2}\right] \nu \left[-\frac{z_1}{z_2}\right] R\left[-z_1\right]$$

If we now take a look at the whole relation two R operaters are next to each other and can be combined right away into one:

$$S = Q\left[\frac{z_1 + z_2}{z_2^2}\right] \nu \left[-\frac{z_1}{z_2}\right] R\left[d - z_1\right] Q\left[\frac{1}{z_1 - d}\right]$$

If we now take another equation into account, which basicly describes, that the Fresnel diffraction operation is equal to premultiplication by a quadratic-phase exponential, an exact scaled Fourier Transform and postmultiplication by a quadratic-phase exponential, we can continue transforming our current equation. But first, the equation describing such a Fresnel diffraction:

$$R\left[d\right] = Q\left[\frac{1}{d}\right] \nu \left[\frac{1}{\lambda d}\right] F Q\left[\frac{1}{d}\right]$$

Taking it into account with our equation this leads us to:

$$R\left[d - z_1\right] = Q\left[\frac{1}{d - z_1}\right] \nu \left[\frac{1}{\lambda\left(d - z_1\right)}\right] F Q\left[\frac{1}{d - z_1}\right]$$

We can now do the actual substitution and get a new operator sequence stating:

$$S = Q\left[\frac{z_1 + z_2}{z_2^2}\right] \nu \left[-\frac{z_1}{z_2}\right] Q\left[\frac{1}{d - z_1}\right] \nu \left[\frac{1}{\lambda\left(d - z_1\right)}\right] F$$

On the last two Q operators we applied $Q\left[c_2\right] Q\left[c_1\right] = Q\left[c_2 + c_1\right]$ and they annihilated each other. Now we need the following relation to flip the Q and ν Operators:

$$Q\left[c\right] \nu\left[t\right] = \nu\left[t\right] Q\left[\frac{c}{t^2}\right]$$

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Applying this equation to the middle part of our equation, we will get two adjacent Q and ν operators:

$$S = Q\left[\frac{z_1 + z_2}{z_2^2}\right] Q\left[\frac{z_1^2}{z_2^2\left(d - z_1\right)}\right] \nu\left[-\frac{z_1}{z_2}\right] \nu\left[\frac{1}{\lambda\left(d - z_1\right)}\right] F$$

$$S = Q\left[\frac{d\left(z_1 + z_2\right) - z_1^2 - z_1 z_2 + z_1^2}{z_2^2\left(d - z_1\right)}\right] \nu\left[\frac{z_1}{-1\lambda\left(d - z_1\right)}\right] F$$

$$S = Q\left[\frac{d\left(z_1 + z_2\right) - z_1 z_2}{z_2^2\left(d - z_1\right)}\right] \nu\left[\frac{z_1}{\lambda\left(z_1 - d\right)}\right] F$$

If we put this now in a more conventional manner concerning the relationship between the input fiel $U_1\left(\xi\right)$ and the output field $U_2\left(\xi\right)$, we get:

$$U_2\left(u\right) = \frac{\exp\left[j\frac{k}{2}\frac{\left(z_1+z_2\right)d - z_1 z_2}{z_2^2\left(d-z_1\right)}u^2\right]}{\sqrt{\frac{\lambda z_2\left(z_1-d\right)}{z_1}}} \int_{-\infty}^{\infty} U_1\left(\xi\right) \exp\left[-j\frac{2\pi z_1}{\lambda z_2\left(z_1 - d\right)}\right] d\xi$$

Which obviously means, that we get a Fourier transform of the input amplitude distribution and along with it a phase factor. This basicly means, a lens always seems to perform a Fourier transformation, even if the fourier plane is not in the focal plane and the object is not hit by collimated, but deverging light. Assuming that $d = \frac{z_1 z_2}{z_1+z_2}$ the phase factor (the first part of the equation) will be transformed to:

$$\frac{1}{\sqrt{\lambda\left(z_1 + z_2\right)}}$$

Which in turn means solely a scaling and therefore we get a full Fourier transform.

Scenario 3:

We now put our object in Distance d behind the lens, z_1 and z_2 are the same as before.

This leads us to the following sequence of Operators:

$$S = R\left[z_2 - d\right] R\left[d\right] Q\left[-\frac{1}{f}\right] Q\left[\frac{1}{z_1}\right]$$

Taking the adjacent Operators together will lead to:

$$S = R\left[z_2\right] Q\left[\frac{1}{z_1} - \frac{1}{f}\right]$$

Doing the formerly used substitution leads us to:

$$S = Q\left[\frac{1}{z_2}\right] \nu\left[\frac{1}{\lambda z_2}\right] F Q\left[\frac{1}{z_2}\right] Q\left[\frac{1}{z_1} - \frac{1}{f}\right]$$

If we take in account the lens equation this simply leads us to:

$$Q\left[\frac{1}{z_2}\right]\nu\left[\frac{1}{\lambda z_2}\right]F$$

Since the operator sequence seemed to be wrong, we will start over with the correct Operator sequence again:

$$S = R\left[d\right]Q\left[\frac{1}{z_2}-\frac{1}{d}\right]Q\left[-\frac{1}{f}\right]Q\left[\frac{1}{z_1}\right]$$

Gathering up all the adjacent Qs we get to:

$$S = R\left[d\right]Q\left[\frac{1}{z_2}-\frac{1}{d}-\frac{1}{f}+\frac{1}{z_1}\right]$$

Taking now the lens law in account, this simplifies to:

$$S = R\left[d\right]Q\left[-\frac{1}{d}\right]$$

We will now apply the same equation to $R\left[d\right]$ as in scenarion 2 and get:

$$S = Q\left[\frac{1}{d}\right]\nu\left[\frac{1}{\lambda d}\right]FQ\left[\frac{1}{d}\right]Q\left[-\frac{1}{d}\right]$$

We can eliminate the Qs in the end, which leads finally to:

$$S = Q\left[\frac{1}{d}\right]\nu\left[\frac{1}{\lambda d}\right]F$$

In this Operatorsequence we can see a foruier transform in the end, before that a scaling (ν) and a phase factor again. In a conventional fashion the phase-factor looks like this:

$$e^{j\frac{k}{2d}x^2}U\left(x\right)$$

It seems impossible to get a full Fourier transform this way, because we always have a phasefactor, we cannot get rid of.

2 Experimenting

We will now describe the experiments, which took place and what our finding were, as well as some interpretation of them.

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2.1 Exp. 1: Collimated beam and variable width slit.

The first experiment used a collimated beam and a variable slit.

The slit in the object plane is transformed to a diffraction pattern in the distant image plane. This diffraction pattern contains information about the slit in a form in which smaller spatial details (narrower slits) are transformed into larger spatial displacement in the image plane (broader diffraction patterns).

Smaller spatial detail can be referred to as a higher "spatial frequency", and the diffraction pattern produces a plot in which greater distance from the optic axis implies greater spatial frequency.

2.2 Exp. 2: Image with few details (newspaper print).

The second experiment featured an image with few details (newspaper print).

The fourier-transformation was used to filter the lower spatial frequencies and thus to reveal more details in the image.

To achieve this, a wire was put into the fourier plane, covering the center in the symmetrical fourier image. In the resulting image the visibility of the newsprint dots was very low, the details or actually edges of the image[5] were quite easy to recognize.

2.3 Exp. 3: Deverging instead of collimated Beam.

In the third experiment, the object (slit) was placed directly in front of a lens and lit with a spreaded laser beam.

We observed a broaded interference pattern compared to the first configuration with a collimated beam.

2.4 Exp. 4: Pattern Recognition.

The fourth experiment featured a kind of pattern recognition, using the configuration with the collimated beam of experiment 1.

The object which contained the pattern was placed in the object plane. A complex filter[6] (similar to a holographic record) was placed in the fourier plane and aligned to the fourier image of the object.

Aside of the image we observed a bright spot. The spot was visible at an angle which corresponded to the angle of the reference beam which was used to create the complex filter.

The pattern was successfully recognized.

[5] a nice lil' doggy
[6] Actually a van der Lught Filter

3 Van der Lugt Filters

Van der Lugt filters are "complex filters"(Changing Intensity as well as phase of passing light). They are created using a holographic like setup with a reference beam and an additional object beam.

The first optical implementation of matched spatial filtering was carried out holographically by Vander Lugt. The spectrum of a target object $h(x, y)$ is caused to interfere with an off axis reference beam in the Fourier Plane and the resulting interference pattern recorded on photographic emulsion. Once the emulsion is developed, it may be reinserted into the Fourier plane to act as a matched spatial filter. If the input object is replaced by another of transmittance $g(x, y)$, illuminated by a plane wave and the reference beam is removed, it can be shown that three beams emerge from the far side of the holographic filter. One beam emerges at zero angle to the optical system and is focussed to the origin of the image plane, but contains no extractable information. Two further beams emerge at equal but opposite angles from the optical axis of the system and are again focussed to the image plane. One contains the convolution of the target function $g(x, y)$ with the object function $h(x, y)$ and the other contains the cross-correlation of the two functions. Assuming the prior mentioned amplitude distribution h(x,y) is Fourier transformed by a lens, it yields the amplitude distribution $\frac{1}{\lambda f} H\left(\frac{x_2}{\lambda f}, \frac{y_2}{\lambda f}\right)$. If a reference beam interferes with this distribution in the Fourier plane under an angle θ, we get the following transmission function for the filter, where $\alpha = \frac{sin\ \theta}{\lambda}$:

$$t_A\left(x_2, y_2\right) \alpha r_0^2 + \frac{1}{\lambda^2 f^2}\left|H\right|^2 + \frac{r_0}{\lambda f} H e^{j2\pi\alpha y_2} + \frac{r_0}{\lambda f} H^* e^{-j2\pi\alpha y_2}$$

Filtering $g(x_1, y_1)$ with our prio described filter, will give the following amplitude distribution, if the light, transmitted by the object hits the filter:

$$U_2 \alpha \frac{r_o^2 G}{\lambda f} + \frac{1}{\lambda^3 f^3}\left|H\right|^2 G + + \frac{r_0}{\lambda^2 f^2} H G e^{j2\pi\alpha y_2} + \frac{r_0}{\lambda^2 f^2} H^* G e^{-j2\pi\alpha y_2}$$

When we now Fourier transform U_2, we get to:

$$U_3\left(x_3, y_3\right) \alpha r_0^2 g\left(x_2, y_2\right) + \frac{1}{\lambda^2 f^2}\left[h\left(x_3, y_3\right) \otimes h^*\left(-x_3, -y_3\right) \otimes g\left(x_3, y_3\right)\right] +$$

$$\frac{r_0}{\lambda f}\left[h\left(x_3, y_3\right) \otimes g\left(x_3, y_3\right) \otimes \delta\left(x_3, y_3 + \alpha\lambda f\right)\right] +$$

$$\frac{r_0}{\lambda f}\left[h^*\left(-x_3, -y_3\right) \otimes g\left(x_3, y_3\right) \otimes \delta\left(x_3, y_3 - \alpha\lambda f\right)\right]$$

The third term is the convolution, the fourth the cross correlation. The cross-correlation describes how big the similarity between the filter and the object is, the convolution in kinda hard to describe in a simple way. since it is pretty

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much only a multiplikation in the frequency plane it kinda describes how much two pictures, maybe we should imagine just two simple figures, a circle and a triangle for example, just black and white, intersect with each other. Assuming both are on top of each other if you align them and the circle is bigger than the triangle, in the resulting picture, you should see a triangle in maximum (black for example) and around it different levels of grey. I am not 100% sure about this trivial description, but I think it roughly describes what is going on. I made some more considerations and must state this is not perfectly true, assuming we have two equal squares at the same position in b&w an align them, we will get a maximal exposed point in the middle and the level of exposure decreases linear to the edges, giving us some sort of pyramid. If we move one square away into on direction, the point of maximum exposure moves halfway into that direction, the level decreases and the pyramid kinda gets stretched into that direction (whereas the "high spot" moves halway into the direction and decreases it's intensity). So it is pretty hard to imagine or describe what is happening, if the the pictures are totally different. The convolution kinda 'mixes' the intensity levels of both pictures (as in finding similarity or overlapping areas) and holds information on how they are shifted against each other. I am sory, but I can't come up with some trivial explanation or description, of what it really does.

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